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# Water waves and Korteweg–de Vries equations

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The classical problem of water waves on an incompressible irrotational flow is considered. By introducing an appropriate non-dimensionalization, we derive four Korteweg–de Vries equations: two expressed in Cartesian co-ordinates and two in plane polars. The equations are: the classical (plane) KdV equation, the two-dimensional 'nearly-plane' equation, the concentric equation and a new 'nearly-concentric' equation. On the basis of the underlying water-wave equations, it is seen that two simple transformations exist between these KdV equations.

By constructing appropriate asymptotic regions defined in terms of the relevant small parameters, we show how various initial value problems give rise to certain solutions of the KdV equations. In particular, the generation of the similarity solutions is examined in detail and it is found that these solutions must eventually match to a solution of the full water-wave equations in a neighbourhood of the origin.

## 1. Introduction

In the last two or three years, the interest that was originally centred around the Korteweg–de Vries (KdV) equation,

$$u_t + uu_x + u_{xxx} = 0,\dagger \tag{1.1}$$

has broadened to the consideration of similar equations in higher dimensions and different co-ordinate systems. Of course, the inverse scattering transform has also played an important role in this extension, but rather by starting from a generalization of the solution technique itself. Armed with a new approach, it is then possible to seek equations which can be solved and which, it is to be hoped, will also prove to be useful. Here, we are particularly interested in the various 'Korteweg–de Vries' equations that arise from the classical water-wave problem. By deriving all the equations – we discuss four – from the same type of problem, it is possible to examine, for example, transformations between the equations. Further, we can also discuss the forms of initial data required to produce specific solutions of the equations. In this way we hope to present a fairly comprehensive account of the role of weak nonlinearity and dispersion in water waves, for various co-ordinate configurations.

In 1895, Korteweg & de Vries gave the first derivation of the equation named after them, essentially by balancing weak nonlinearity against weak (linear) dispersion. Actually, this idea was not new, for the solitary wave had been the subject of discussion for quite a few years (see Russell 1844; Boussinesq 1871; Rayleigh 1876). However, the

† We express the equations in a simple form in the introduction: a specific interpretation is not implied by the symbols. Throughout, subscripts denote partial derivatives.

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amazing properties of the classical KdV equation remained unnoticed until the early fifties when some related numerical work at Los Alamos produced rather surprising results (Fermi, Pasta & Ulam 1955). This led eventually to the pioneering work by the group at Princeton in the middle sixties (e.g. Gardner *et al.* 1974), after which the inverse scattering transform became well-established with very wide applications and generalizations. Once the KdV equation (and other equations) in one space dimension were understood and solved, the search for equations in two (or more) space dimensions began. For example, the two-dimensional KdV equation,

$$(u_t + uu_x + u_{xxx})_x + u_{yy} = 0, \tag{1.2}$$

first seems to have appeared in 1970 (Kadomtsev & Petviashvili), and the inverse scattering transform for this equation is due to Dryuma (1974) (see also Zakharov & Shabat 1974). (Since the $y$ dependence in the original problem must be 'weak' for this equation to be valid, as we shall see, we prefer to call it the 'nearly-plane' KdV equation.)

The equation for purely concentric waves was first written down by Maxon & Viecelli (1974),

$$2u_r + u/r + uu_x + u_{xxx} = 0, \tag{1.3}$$

and discussed at some length by Miles (1978*a*) in the context of the water-wave problem. A general inverse-scattering technique for the concentric KdV equation has been developed by Calogero & Degasperis (1978), but details of specific solutions are not given (and construction of similarity solutions is not straightforward). In fact, an inverse transform can be obtained directly from that for the nearly-plane KdV equation (Johnson 1979), and some special results for the nearly-plane equation appear in the work of Johnson & Thompson (1978); we shall discuss the connexion between nearly-plane and concentric KdV equations in due course. As the plane KdV equation can be extended to the nearly-plane, so the concentric equation can be generalized to

$$\left(2u_r + \frac{1}{r}u + uu_x + u_{xxx}\right)_x + \frac{1}{r^2}u_{\theta\theta} = 0. \tag{1.4}$$

This new equation we shall call the nearly-concentric KdV equation, in accord with our previous convention; the derivation will be given later.

In this paper we shall outline the derivation of these four KdV equations, paying particular attention to the scalings required to produce each one. These scalings suggest the existence of transformations between the equations, and this is confirmed. Also, in order to relate various solutions to specific initial data, we discuss the form of the matching problem for small times. This proves to be rather elementary for one class of solutions, but not for the similarity solutions. The matching of the similarity solutions is considered in some detail, placing emphasis on the various regions of validity (measured in terms of the relevant small parameter). This, therefore, extends the work of Miles (1978*a*) and, further, we match for arbitrary amplitude of the similarity solutions. (A careful examination of the calculation due to Miles shows that matching was performed only for the amplitude tending to zero.)

The similarity solutions used here are not all new: see, for example, Berezin & Karpman (1964); Miles (1978*a*, *b*); Rosales (1978); Ablowitz & Segur (1977*a*). However, for the concentric problem we find two similarity solutions both of which are matched to an appropriate linear régime. One is just the classical form given by the Painlevé equation, and used by Miles (1978*a*); the second is a new solution which can be expressed in closed form and was obtained directly from the inverse scattering transform (Johnson & Thompson 1978). We find that, in terms of the linear matched problem, there is a very simple relation between the two similarity solutions.

## 2. Derivation of the KdV equations

We start from the classical water-wave problem that is, an incompressible irrotational fluid bounded above by a free surface and below by a rigid horizontal surface. The fluid extends to infinity in all horizontal directions and the free surface is characterized by the simplest of conditions, namely constant pressure. In the absence of any disturbances, the fluid will be stationary with a constant depth $d$. To write down an appropriate non-dimensional problem, it is convenient to introduce a typical horizontal scale, $l$, which may be interpreted as a wavelength of the surface disturbance. Thus, using $d$, $l$, a typical amplitude '$a$' and a speed $(gd)^{\frac{1}{2}}$, where $g$ is the acceleration of gravity, the equations and boundary conditions become

$$\frac{\partial^2\phi}{\partial z^2}+\delta^2\nabla_\perp^2\phi = 0, \quad \frac{\partial\phi}{\partial z} = 0 \quad \text{on} \quad z = 0, \tag{2.1$a,b$}$$

$$\left.\begin{aligned} \eta+\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\alpha\left[\frac{1}{\delta^2}\left(\frac{\partial\phi}{\partial z}\right)^2+(\nabla_\perp\phi)^2\right] &= 0 \quad (2.2)\\ \frac{\partial\phi}{\partial z} = \delta^2\left[\frac{\partial\eta}{\partial t}+\alpha(\nabla_\perp\phi).(\nabla_\perp\eta)\right] & \quad (2.3) \end{aligned}\right\} \quad \text{on} \quad z = 1+\alpha\eta,$$

when written in the non-dimensional form. The vertical co-ordinate measured up from the bottom of the fluid is $z$, and the free surface is at $z = 1+\alpha\eta$. The gradient operator perpendicular to $z$ is represented by $\nabla_\perp$, so that we may choose

$$\nabla_\perp \equiv \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) \quad \text{or} \quad \nabla_\perp \equiv \left(\frac{\partial}{\partial r},\frac{1}{r}\frac{\partial}{\partial\theta}\right),$$

for example. The parameters appearing in equations (2.1)–(2.3) are defined by

$$\alpha = a/d, \quad \delta = d/l, \tag{2.4}$$

whence $\alpha$ is the amplitude parameter and $\delta$ the long-wave (or shallowness) parameter.

The linear problem corresponding to equations (2.1)–(2.3) is obtained simply by setting $\alpha = 0$: this retains full (linear) dispersion, but for weak dispersion ($\delta\to 0$) where $\partial\phi/\partial z = O(\delta^2)$, we find that

$$\frac{\partial^2\eta}{\partial t^2}-\nabla_\perp^2\eta = 0. \tag{2.5}$$

In other words, small amplitude long waves satisfy the general wave equation for propagation on a surface. However, this equation is valid only for sufficiently short times: eventually nonlinear and dispersive terms cannot be neglected even when both $\alpha$ and $\delta$ are small. The simplest approach which enables these other effects to be examined is to choose $\delta^2 = O(\alpha)$ and consider waves travelling only in one direction. Hence for plane waves we write

$$\nabla_\perp \equiv \left(\frac{\partial}{\partial x},O\right), \quad \xi = x-t, \quad \tau = \alpha t \tag{2.6}$$

and then equations (2.1)–(2.3) yield

$$2\eta_\tau + 3\eta\eta_\xi + \tfrac{1}{3}(\delta^2/\alpha)\,\eta_{\xi\xi\xi} = 0 \tag{2.7}$$

to leading order. The steady-state form of this equation is essentially as derived by Korteweg & de Vries (1895) for their 'cnoidal' waves. Throughout, we shall restrict ourselves to the study of waves travelling in one direction only: an initial profile on compact support will eventually travel in opposite directions as two independent wave groups.

To produce an equation which is predominantly a balance between weak non-linearity and weak dispersion (in the direction of propagation), and which is also two dimensional, presupposes a nearly-plane wave. If we choose

$$\nabla_\perp \equiv \left(\frac{\partial}{\partial x}, \ \frac{\partial}{\partial y}\right)$$

and introduce the general scaling†

$$\left.\begin{aligned} \xi = \frac{\alpha^{\frac{1}{2}}}{\delta}(x-t), \quad \tau = \frac{\alpha^{\frac{3}{2}}}{\delta}t, \quad \mu = \frac{\alpha}{\delta}y,\\ \text{with} \qquad \Phi = \frac{\alpha^{\frac{1}{2}}}{\delta}\phi, \qquad \end{aligned}\right\} \tag{2.8}$$

then the leading order gives

$$2\eta_\tau + 3\eta\eta_\xi + \tfrac{1}{3}\eta_{\xi\xi\xi} + \Phi_{\mu\mu} = 0, \quad \Phi_\xi = \eta. \tag{2.9}$$

Equations (2.9), which are equivalent to the single equation

$$(2\eta_\tau + 3\eta\eta_\xi + \tfrac{1}{3}\eta_{\xi\xi\xi})_\xi + \eta_{\mu\mu} = 0, \tag{2.10}$$

are valid for arbitrary $\delta$ provided $\alpha \to 0$. Clearly, a special case of (2.8) is $\delta^2 = O(\alpha)$ for which $\xi = O(1)$ and $\phi = O(1)$; similarly, a special case of (2.10) is the (classical) plane KdV equation. As already indicated, we shall refer to (2.10) as the nearly-plane KdV equation.

The problem of concentric waves is rather more complicated for the amplitude decays like $r^{-\frac{1}{2}}$, where $r$ is the radius of the wave front. Thus to obtain the appropriate balance for a KdV equation the amplitude must be scaled: this is to be compared with the plane (and nearly-plane) equations for which $\eta = O(\alpha)$ uniformly for all times. In polar co-ordinates, we introduce

$$\nabla_\perp \equiv \left(\frac{\partial}{\partial r}, 0\right),$$

and write

$$\xi = \frac{\alpha^2}{\delta^2}(r-t), \quad \tau = \frac{\alpha^6}{\delta^4}t, \tag{2.11}$$

with

$$\Phi = \alpha^{-1}\phi, \quad H = \frac{\delta^2}{\alpha^3}\eta. \tag{2.12}$$

† We find it convenient to use the same symbols to represent the linear characteristic, $\xi$, and long time variable, $\tau$, in all geometries.

Expanding in powers of $\Delta = \alpha^4/\delta^2$, and expressing

$$\frac{\alpha^6}{\delta^4} r = \tau + \Delta\xi,$$

the leading order from equations (2.1)–(2.3) yields

$$2H_\tau + \frac{1}{\tau} H + 3HH_\xi + \tfrac{1}{3}H_{\xi\xi\xi} = 0. \tag{2.13}$$

Equation (2.13) is the concentric KdV equation and is valid for arbitrary $\alpha$, $\delta$ provided only that $\Delta \to 0$. (The amplitude parameter, $\alpha$, must now be interpreted as based specifically on the amplitude of the wave when $r = O(1)$, $t = O(1)$.)

An equation for nearly-concentric waves can be found in a similar manner to that employed for the nearly-plane equation. We now introduce

$$\nabla_\perp = \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}\right),$$

with the definitions given in (2.11) and (2.12), and noting that for the nearly-plane KdV equation the angle subtended by the wave front at the origin is $O(\alpha^{\frac{1}{2}})$, we introduce

$$\psi = \theta/\Delta^{\frac{1}{2}}, \tag{2.14}$$

then as $\Delta \to 0$

$$\left(2H_\tau + \frac{1}{\tau} H + 3HH_\xi + \tfrac{1}{3}H_{\xi\xi\xi}\right)_\xi + \frac{1}{\tau^2} H_{\psi\psi} = 0, \tag{2.15}$$

to leading order. Thus the angle subtended in both the nearly-plane and 'nearly-concentric' equations is small, and essentially identical, since $\Delta$ here replaces $\alpha$ in the derivation of equation (2.10); it is in this sense that we have dubbed equation (2.15) the nearly-concentric equation.

## 3. Transformations

Since we have used simply Cartesian or polar co-ordinates, then

$$r^2 = x^2 + y^2, \quad \tan\theta = y/x.$$

Hence for a nearly-plane wave front,

$$r - t = x(1 + \tfrac{1}{2}y^2/x^2 + \dots) - t,$$

but also $x \sim t$ for large $t$ when $x - t = O(1)$ and so, alternatively,

$$r - t = x - t + \tfrac{1}{2}y^2/t + \dots .$$

If we now introduce the scaled variables given in (2.8) we obtain

$$r - t = \frac{\delta}{\alpha^{\frac{1}{2}}}\left[\xi + \frac{1}{2}\frac{\mu^2}{\tau} + \dots\right],$$

which relates the co-ordinates used in the nearly-plane equation with those in the concentric KdV equation. Of course, this transformation is only approximate but it does suggest that we rewrite the nearly-plane equation, (2.10), using the form†

$$\eta = \eta(\tau, \hat{\xi}), \quad \hat{\xi} = \xi + \tfrac{1}{2}\mu^2/\tau, \tag{3.1}$$

and then we find

$$2\eta_\tau + \frac{1}{\tau}\eta + 3\eta\eta_{\hat{\xi}} + \tfrac{1}{3}\eta_{\hat{\xi}\hat{\xi}\hat{\xi}} = 0, \tag{3.2}$$

after one integration in $\hat{\xi}$. Further, equation (3.2) is invariant under the transformation

$$\tau \to \frac{\delta^3}{\alpha^{\frac{9}{2}}}\tau, \quad \hat{\xi} \to \frac{\delta}{\alpha^{\frac{3}{2}}}\hat{\xi}, \quad \eta \to \frac{\alpha^3}{\delta^2}\eta,$$

which therefore produces exactly the parameter dependence required for the concentric KdV equation, (2.13). In other words, the change of variable given in (3.1) is exactly the distortion necessary in the far field to produce a nonlinear dispersive concentric wave.

On the basis of this result, we can anticipate that there will be a corresponding choice of angular dependence, $\psi$, in the nearly-concentric equation which enables the plane wave to be recovered. Following the same approach, we write

$$x - t = r - t - \tfrac{1}{2}t\theta^2 + \ldots = \frac{\delta^2}{\alpha^2}[\xi - \tfrac{1}{2}\tau\psi^2 + \ldots],$$

which suggests the change of variable

$$H = H(\tau, \xi), \quad \hat{\xi} = \xi - \tfrac{1}{2}\tau\psi^2 \tag{3.3}$$

in equation (2.15); this yields

$$2H_\tau + 3HH_{\hat{\xi}} + \tfrac{1}{3}H_{\hat{\xi}\hat{\xi}\hat{\xi}} = 0.$$

Thus, although the four KdV equations can be expected to apply to four different types of initial data, under special conditions the four equations reduce to just two. One special case that we shall examine rather carefully is that which occurs when similarity solutions are constructed.

## 4. Simple matching

Before we turn to the more involved question of similarity solutions and their matching, we deal with the matching problem for $\tau \to 0$ which involves functions essentially independent of $\tau$ (although the nearly-concentric equation is exceptional in this respect). For example, in the case of the plane KdV equation (2.7) – or from (2.9) – this enables $\eta(\xi, \tau)$ to be matched to $f(\xi)$ as $\tau \to 0$. More formally, if we introduce the near-field variables

$$X = \frac{\alpha^{\frac{1}{2}}}{\delta}x, \quad T = \frac{\alpha^{\frac{1}{2}}}{\delta}t, \quad \Phi = \frac{\alpha^{\frac{1}{2}}}{\delta}\phi, \tag{4.1}$$

then to leading order as $\alpha \to 0$

$$\eta_{TT} - \eta_{XX} = 0.$$

† Of course, if we include $\hat{\psi} = \mu/\tau$ in (3.1) then we obtain the nearly-concentric KdV equation; similarly, $\hat{\mu} = \psi\tau$ in (3.3) yields the nearly-plane equation.

Choosing right-running waves, we have the solution

$$\eta(X,T) = f(X-T) = f(\xi)$$

for arbitrary $f(\xi)$ (although for a number of reasons we would normally require $f(\xi) \to 0$ sufficiently rapidly as $|\xi| \to \infty$). The matching is therefore between the near-field (as $T \to \infty$) and the far-field (as $\tau \to 0$), and the net result is the posing of an initial value problem for the KdV equation.

The corresponding argument for the nearly-plane equation is almost identical, requiring only the addition of the $y$ variable in the form $\mu = \alpha y/\delta$. To leading order, the one-dimensional wave equation still pertains so that

$$\eta = f(X-T, \mu) = f(\xi, \mu)$$

in the near field. Again, $f$ is arbitrary but now it may be a function of $\mu$; this, of course, describes a nearly-plane wave for the $y$ dependence is $O(\alpha^{\frac{1}{2}})$ smaller than the dependence on $\xi$.

The matching problem for concentric waves is also rather straightforward provided it is noted that $\phi$ and $\eta$ must be scaled to account for the geometric effect. Thus in the near field we make use of the variables

$$R = \frac{\alpha^2}{\delta^2} r, \quad T = \frac{\alpha^2}{\delta^2} t, \quad h = \frac{\delta}{\alpha} \eta, \quad \hat{\phi} = \frac{\alpha}{\delta} \phi, \tag{4.2}$$

whence the full problem yields

$$h_{TT} - \left(h_{RR} + \frac{1}{R} h_R\right) = 0 \tag{4.3}$$

to leading order as $\Delta = \alpha^4/\delta^2 \to 0$. The solution of the concentric KdV equation can then be matched to a solution, $h(R,T)$, of (4.3) which satisfies

$$h(R,T) \sim \frac{1}{R^{\frac{1}{2}}} f(R-T) \sim \frac{1}{T^{\frac{1}{2}}} f(\xi) \tag{4.4}$$

as $R, T \to \infty$ (for $R-T$ fixed). The form of (4.4), for arbitrary $f(\xi)$, therefore presents the KdV equation with an appropriate initial value problem. Finally, we consider the nearly-concentric KdV equation (2.15) and this turns out to be a little more difficult to analyse in terms of a matching problem. That this should be the case is hardly surprising when the form of (2.15) is examined for $\tau \to 0$: it is clear that either the solution is independent of $\psi$ (and then we have the concentric equation) or the terms involving $\psi$ are exponentially small, as $\tau \to 0$. In the near field, using the variables $R$, $T$ and $\psi$, the appropriate expansion which matches takes the form

$$\hat{\phi} \sim \sum_{m=0}^{\infty} e^{-\lambda_m/\rho} \left( \sum_{n=0}^{\infty} \Delta^n \hat{\phi}_{mn}(R,T,z,\psi) \right),$$

where $\rho = \Delta R$. The constants $\lambda_m$ are chosen to describe specific initial data, but such that $\lambda_0 = 0$ and $\lambda_m > 0$ $(m > 0)$; also $\hat{\phi}_{0n} = \hat{\phi}_{0n}(R,H,z)$ $(n \geqslant 0)$. If $\hat{\phi}_{0n} \not\equiv 0$, then the solution is predominantly concentric and the matching to the leading-order term yields a concentric initial value problem for (2.15). On the other hand, if $\hat{\phi}_{0n} \equiv 0$ (all $n$) the solution involves $\psi$ and grows exponentially as $R$ increases allowing matching when $R-T = \xi = O(1)$, $T = O(\Delta^{-1})$, $h = O(\Delta^{\frac{1}{2}})$, $\hat{\phi} = O(\Delta^{\frac{1}{2}})$. The details of the

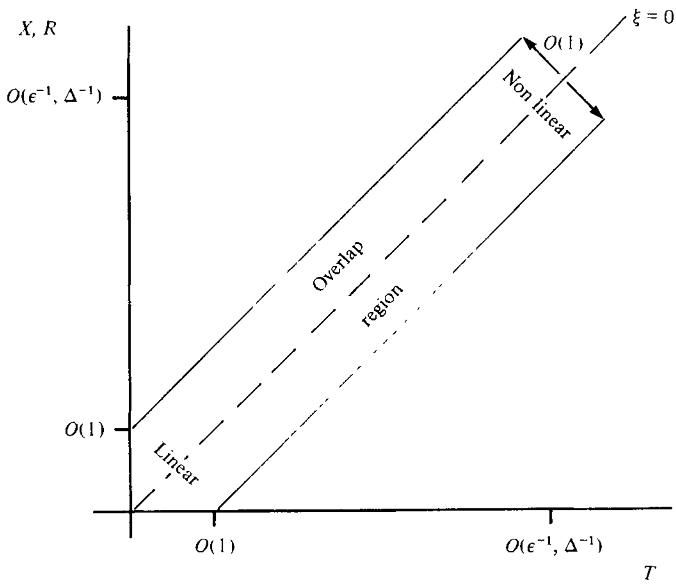


FIGURE 1. Regions of validity for the plane and concentric KdV equations: simple matching.

near-field solution – and its matching – are rather intricate and since they apply to very special forms of initial data we prefer to relegate this problem to future study.

Finally, because we need them for comparison later, let us sketch the regions of validity for the KdV equations and their near-field counterparts. Limiting ourselves to the $x, t$ (or $r, t$) plane, which is all that is necessary for the similarity solutions, we obtain the very simple sketch given in figure 1. The asymptotic regions for the plane KdV equation ($X, T$ and scale $\alpha^{-1}$) are seen to be identical to those for the concentric equation ($R, T$ and scale $\Delta^{-1}$) when described in this way (that is, for solutions with $\xi = O(1)$ uniformly). The matching occurs in the overlap region described by $\xi = O(1)$ and $T \to \infty$, $\tau \to 0$ (where $\tau = \alpha T$ or $\Delta T$).

## 5. Similarity solutions: derivation

We have already obtained the elementary transformations that take

nearly-plane $\to$ concentric,

nearly-concentric $\to$ plane,

and we shall consider only the plane and concentric KdV equations hereafter, together with their appropriate similarity solutions. It will soon become evident that the most interesting problem arises with the concentric equation, (2.13). This is the equation discussed by Miles (1978*a*), and also the one which can be analysed in the greatest detail: we therefore begin our discussion with (2.13).

The similarity solution can be expressed in the form

$$H(\tau, \xi) = \tau^{-\frac{2}{3}} F(\zeta), \quad \zeta = \xi \tau^{-\frac{1}{3}}, \tag{5.1}$$

where

$$-\tfrac{1}{3}F - \tfrac{2}{3}\zeta F' + 3FF' + \tfrac{1}{3}F''' = 0, \tag{5.2}$$

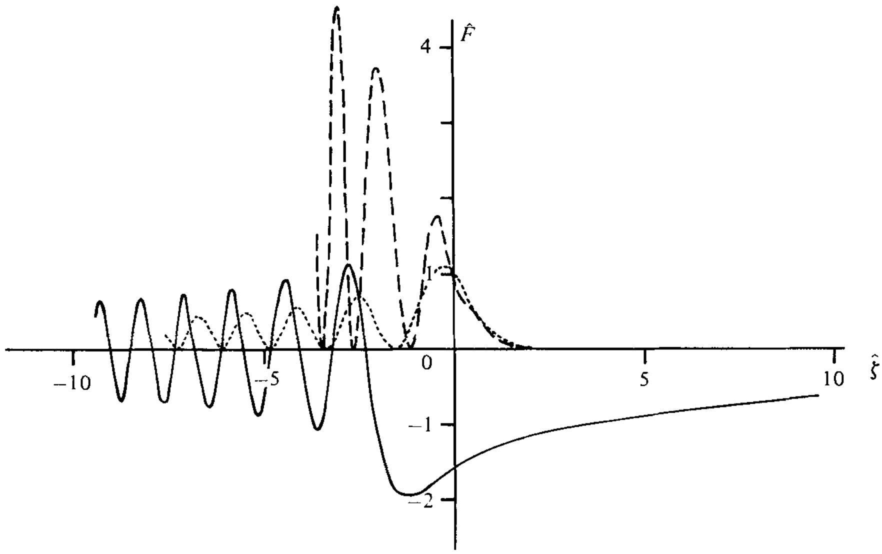


FIGURE 2. Similarity solutions: from equation (5.3) with $A = 0$ and $\hat{F}(0) = 1$ (....); from equation (5.3) with $A = -8$ (——); from equation (5.14) for $\hat{F} = (\frac{3}{2})^{\frac{1}{3}} F$ against $\hat{\zeta}$ with $\hat{F}(0) = 1$ (– – –).

and the primes denote derivatives with respect to $\zeta$. Upon multiplication by $F$ and integrating once we obtain

$$\hat{F}\hat{F}'' - \tfrac{1}{2}\hat{F}'^2 + 2(\hat{F}^3 - \zeta\hat{F}^2) = A, \tag{5.3}$$

where the convenient transformation

$$\zeta = 2^{\frac{1}{3}}\zeta, \quad F = (\tfrac{2}{3})^{\frac{1}{3}}\hat{F} \tag{5.4}$$

has been employed. In (5.3), $A$ is the arbitrary constant of integration and if we require the solution for which $\hat{F} \to 0$ exponentially then $A = 0$†. The equation is then expressed as

$$v'' + v^3 - \zeta v = 0 \tag{5.5}$$

where $\hat{F} = v^2$: this is a Painlevé equation of the second kind (see Ince 1944; Miles 1978*b*; Rosales 1978). Solutions to equation (5.5), and hence to (5.3) for $A = 0$, can be obtained quite easily by numerical integration and one such solution is given in figure 2. Of particular interest here are the asymptotic behaviours as $\zeta \to \pm\infty$ which are obtained from the (linear) Airy equation‡ as

$$H(\tau,\xi) \sim \begin{cases} \dfrac{a}{\pi}(-\xi\tau)^{-\frac{1}{2}}\sin^2\left[\dfrac{\sqrt{2}}{3}(-\xi\tau^{-\frac{1}{3}})^{\frac{3}{2}} + \dfrac{3a}{4\pi}\ln|\xi\tau^{-\frac{1}{3}}| + \dfrac{\pi}{4}\right], & \xi < 0, \quad (5.6) \\ \dfrac{b}{4\pi}(\xi\tau)^{-\frac{1}{2}}\exp\left[\dfrac{-2\sqrt{2}}{3}(\xi\tau^{-\frac{1}{3}})^{\frac{3}{2}}\right], & \xi > 0, \quad (5.7) \end{cases}$$

† Exponential decay of $\hat{F}$ seems the most reasonable (physical) condition to impose, although (5.3) can still be written as a Painlevé–2 equation even for arbitrary $A$.

‡ The logarithmic term in (5.6) arises from higher-order terms, see Miles (1978*b*); Ablowitz & Segur (1977*b*).

and these are valid, for example, as $\tau \to 0$, $\xi$ fixed. The constants $a$ and $b$ are related since the asymptotic behaviours (5.6), (5.7) pertain to the same solution. Ablowitz & Segur (1977*b*; see also Miles 1978*b*) suggest that

$$\hat{a}^2 = \ln(1+\hat{b}^2); \quad \hat{a}^2 = 3\times 2^{-\frac{4}{3}}a, \quad \hat{b}^2 = 3\times 2^{-\frac{4}{3}}b. \tag{5.8}$$

This conjecture clearly satisfies the requirement that, for small amplitude, $a \sim b$ and then the solution is everywhere proportional to $A_i^2(\zeta)$. Further, (5.8) shows that $0 < b < \infty$ for a solution to exist and this has been confirmed by numerical integration both for $v$ and $V = iv$; the equation for $V$ is then (5.16) (see, for example, Berezin & Karpman 1964; Miles 1978*b*; Rosales 1978). We have performed a number of numerical integrations and the estimates for $\hat{a}$ and $\hat{b}$ agree very closely with equation (5.8).

Let us now turn to the formulation of a second similarity solution of the concentric KdV equation. This is obtained directly from the inverse scattering transform for the nearly-plane KdV equation. Of course, it is quite beyond the scope of this paper to give the details but the reader may find a brief outline useful; more information is given in Johnson & Thompson (1978) although the limit eventually used here is new.

The solution of the nearly-plane KdV equation, (2.10), can be represented by the inverse-scattering transform as

$$\eta(\xi,\mu,\tau) = \frac{4}{3}\frac{\partial}{\partial\xi}K(\xi,\xi;\mu,\tau),$$

where

$$K(\xi,z;\mu,\tau)+F(\xi,z;\mu,\tau)+\int_\xi^\infty K(\xi,s;\mu,\tau)\,F(s,z;\mu,\tau)\,ds = 0,$$

with

$$F_{\xi\xi\xi}+F_{zzz}+\tfrac{3}{2}F_\tau = 0, \quad F_{\xi\xi}-F_{zz}+F_\mu = 0.$$

If we choose

$$F(\xi,z;\mu,\tau) = \hat{\alpha}(\xi;\mu,\tau)\,\hat{\beta}(z;\mu,\tau),$$

then

$$\eta = \frac{4}{3}\frac{\partial^2}{\partial\xi^2}\left\{\ln\left[1+\int_\xi^\infty \hat{\alpha}(s;\mu,\tau)\,\hat{\beta}(s;\mu,\tau)\,ds\right]\right\},$$

provided functions $\hat{\alpha}$ and $\hat{\beta}$ can be found. One choice, which retains a similarity form for $F$, is obtained by introducing the similarity variable

$$\zeta = (\xi+\tfrac{1}{2}\mu^2/\tau)\,\tau^{-\frac{1}{3}}$$

(compare with (3.1) and (5.1)), and then

$$\eta = \tfrac{4}{3}\tau^{-\frac{2}{3}}\frac{\partial^2}{\partial\zeta^2}\ln\left[1+k\tau^{-\frac{1}{3}}\int_\zeta^\infty A_i^2(\zeta')\,d\zeta'\right], \tag{5.9}$$

where $k$ is an arbitrary constant. As it stands, (5.9) is not a similarity solution but if we allow $k\to\infty$ then the resulting expression is defined and equivalent to (5.1) yielding

$$\eta = \tfrac{4}{3}\tau^{-\frac{2}{3}}\frac{d^2}{d\zeta^2}\left\{\ln\left(\int_\zeta^\infty A_i^2(\zeta')\,d\zeta'\right)\right\}. \tag{5.10}$$

Comparing (5.10) and (5.1), and noting the transformation between the nearly-plane and concentric equations, we see that this produces a similarity solution of the concentric KdV equation with

$$\hat{F}(\zeta) = 2\frac{d^2}{d\zeta^2}\left\{\ln\left(\int_\zeta^\infty A_i^2(\zeta')\,d\zeta'\right)\right\}. \tag{5.11}$$

Thus we have constructed a second solution to equation (5.3) which is explicit, although it is completely determined (that is, without the freedom of an arbitrary constant). It is easily confirmed that (5.11) is a solution of (5.3) when $A = -8$†. This new solution is oscillatory as $\zeta \to -\infty$ and decays *algebraically* (like $\zeta^{-\frac{1}{2}}$) as $\zeta \to +\infty$: the solution appears on figure 2. To enable the matching to be performed later, we note that

$$H(\tau,\xi) \sim \begin{cases} \dfrac{2\sqrt{2}}{3}(-\xi\tau)^{-\frac{1}{2}}\cos\left[\dfrac{2\sqrt{2}}{3}(-\xi\tau^{-\frac{1}{3}})^{\frac{3}{2}}\right], & \xi < 0, \qquad (5.12)\\[2ex] \dfrac{-2\sqrt{2}}{3}(\xi\tau)^{-\frac{1}{2}}, \quad \xi > 0, & \qquad (5.13)\end{cases}$$

as $|\zeta| \to \infty$, for example $\tau \to 0$, $\xi$ fixed. When the matching problem for the two similarity solutions is developed, we shall be able to indicate the close affinity between the cases $A = 0,\ -8$ [see equation (5.3)].

Finally, we consider the similarity solution of the plane (classical) KdV equation [see equations (2.7), (2.9) or (2.10)]. This problem has been examined before; see, for instance, Berezin & Karpman (1964) and Rosales (1978). The form of the solution is exactly as used in (5.1), resulting in the ordinary differential equation

$$-\tfrac{2}{3}F - \tfrac{1}{3}\zeta F' + 3FF' + \tfrac{1}{3}F''' = 0, \qquad (5.14)$$

where $\zeta = \xi\tau^{-\frac{1}{3}}$. Transformations exist which enable equation (5.14) to be integrated directly: the one we employ here is due to G. B. Whitham and is used in Rosales (1978). Following their lead, let us set

$$F = \lambda\omega' - \omega^2, \qquad (5.15)$$

where $\lambda$ is a constant to be determined. From (5.14) we find that

$$(\tfrac{1}{3}\omega'' - \tfrac{1}{3}\zeta\omega - \omega^3)' = \tilde{\alpha}\exp\left[\frac{2}{\lambda}\int^{\zeta}F(\zeta')\,d\zeta'\right]$$

provided $\lambda^2 = \frac{2}{3}$, where $\tilde{\alpha}$ is an arbitrary constant. For a solution in which $\omega$ (and therefore $F$) decays exponentially, we require

$$V'' - \zeta V - V^3 = 0, \quad \zeta = 2^{-\frac{1}{3}}\zeta, \quad \omega = \frac{2^{\frac{1}{3}}}{\sqrt{3}}V, \qquad (5.16)$$

which is a Painlevé equation of the second kind. (This is related to (5.5) by the transformation $V = iv$.) A typical solution of (5.14), obtained by numerical integration, is given in figure 2 and the asymptotic behaviours (deduced directly from (5.14) or via (5.15)) are

$$\eta(\tau,\xi) \sim \begin{cases} \dfrac{2A}{\sqrt{\pi}}(-\xi\tau^{-3})^{\frac{1}{4}}\cos\left[\dfrac{2\sqrt{2}}{3}(-\xi\tau^{-\frac{1}{3}})^{\frac{3}{2}} + \dfrac{\pi}{4}\right], & \xi < 0; \qquad (5.17)\\[2ex] \dfrac{B}{\sqrt{\pi}}(\xi\tau^{-3})^{\frac{1}{4}}\exp\left[-\dfrac{2\sqrt{2}}{3}(\xi\tau^{-\frac{1}{3}})^{\frac{3}{2}}\right], & \xi > 0. \qquad (5.18)\end{cases}$$

These are valid as $|\zeta| \to \infty$, which we can interpret as $\tau \to 0$, $\xi$ fixed, for the purpose of the matching later. The constants $A$ and $B$ are related by

$$\hat{A}^2 = -\ln(1-\hat{B}^2), \quad \hat{A} = 2^{\frac{3}{4}}A/3, \quad \hat{B} = 2^{\frac{3}{4}}B/3, \qquad (5.19)$$

† This value of $A$ turns out to be equivalent to $\delta = \frac{2}{3}$ in the set of similarity solutions recently found by Airault (1979) for the plane KdV equation.

if the conjecture given in (5.8) is correct: for non-singular solutions of equation (5.16) we require $1 > \hat{B}^2 > 0$ (see Miles 1978*b*; Rosales 1978). The asymptotic behaviour of this solution for $\zeta \to -\infty$ is an oscillation with an amplitude which increases like $|\zeta|^{\frac{1}{4}}$ (see figure 2). Consequently, it is not clear how this behaviour can be matched to a realistic solution in the near field: we might expect some difficulties. Armed with some properties of the similarity solutions, we can now examine the way in which these can be matched to appropriate near-field solutions.

## 6. Similarity solutions: matching

The matching process that we describe here is carefully analysed in terms of the relevant small parameter: $\Delta = \alpha^4/\delta^2$ for the concentric equation (see (2.11)–(2.13)) and $\alpha$ for the plane equation. The problem for the concentric equation, which we shall see can be quite detailed, was first discussed by Miles (1978*a*). Here, we extend the results to show how the matching can be performed between certain well-defined regions and also that it holds for arbitrary (i.e. $O(1)$) amplitudes of the similarity solution†: this latter point leads to quite an involved description in the $R, T$ plane. Further, we are able to match both similarity solutions in the near-field, giving corresponding linear solutions which turn out to differ only slightly. The matching problem for the plane KdV equation, which is not so rewarding, will be given in due course.

The general form of a similarity solution suggests that the solution for matching must be expressed in a suitable multiple-scale representation. To retain the dependence on the similarity variable, $\xi\tau^{-\frac{1}{3}}$, we introduce

$$\chi = \Delta^{-n}\xi, \quad \sigma = \Delta^{-3n}\tau = \Delta^{1-3n}T, \tag{5.20}$$

and seek a solution of the full problem which is a function of $\chi$, $\sigma$ and $T$ (on the surface): this will yield a solution valid essentially for $T = O(1)$. The parameter $n$, in (5.20), enables various regions of the nonlinear problem to be matched, provided only that $\frac{1}{3} < n < \frac{1}{2}$ (which ensures that the appropriate ordering of the terms in the full problem is maintained). A solution which is dependent only on $\chi$ and $\sigma$ is possible, but this leads directly to the concentric KdV equation itself if $0 \leqslant n < \frac{1}{2}$, and then only the similarity solution is available (because of (5.20)). In fact we can even permit $n < 0$ and the KdV equation still follows, but when $n = \frac{1}{2}$ we obtain the full equations (see (2.1)–(2.3)) bereft of any parameters whatsoever. In other words, the similarity solution is valid in a neighbourhood about $\xi = 0$ – which becomes narrower as $T$ decreases – and when $\xi$ and $T$ are $O(\Delta^{\frac{1}{2}})$ the full problem is recovered with all the parameter dependence scaled out. This means, of course, that we shall not be able to give a complete description of the near-field behaviour (where $T = O(\Delta^{\frac{1}{2}})$); this can be compared with the simple matching process used in §4 where $\xi = O(1)$ in all regions. Although the close neighbourhood of the origin in the $R$, $T$ plane must be avoided, we shall demonstrate that matching is possible to regions outside those where the similarity solution is valid. Further, it also follows that the matching ahead and behind the nonlinear solution must be treated as two quite separate problems on either side of the similarity solution.

Now, using the variables (5.20), retaining $T$ and expressing $(|\xi|\tau)^{-\frac{1}{2}}$ (in (5.6) and

† Of course, the amplitude is necessarily fixed for our second similarity solution [given by (5.11)].

(5.7)) in terms of $\chi$ and $T$, we can scale $\eta$ and $\phi$ appropriately. The full problem, (2.1)–(2.3), then yields

$$2\hat{H}_\sigma + \tfrac{1}{3}\hat{H}_{\chi\chi\chi} = 0, \quad H = \Delta^{-\frac{1}{2}(1+n)}\hat{H}, \tag{5.21}$$

to leading order, and higher-order terms give

$$2\hat{H}_T + \frac{1}{T}\hat{H} = 0, \tag{5.22}$$

if we write $R = T + \Delta^n\chi$ in the terms $R^{-1}$, $R^{-2}$ in the original equations. Thus from (5.21) and (5.22) we obtain

$$\hat{H} = T^{-\frac{1}{2}}\int_0^\infty A_\pm(k)\cos\left(k\chi + \frac{k^3}{6}\sigma + \theta_0\right)dk, \tag{5.23}$$

where $A_\pm(k)$ ($\chi \gtrless 0$) must be found by matching (5.23) with (5.6) and (5.7) (see Miles 1978*a*). The matching problem is developed in two stages: first the general form of $A_\pm(k)$ must be determined and then precise constants in $A_\pm$ have to be found.

The matching is most conveniently accomplished by expressing (5.23) in original variables, i.e. $\xi$, $T$, and then consider $\Delta \to 0$. This suggests the change of variable $k = k_1\Delta^{n-\frac{1}{2}}$ so that

$$I = \int_0^\infty A_\pm(k)\,e^{i(k\chi+\frac{1}{6}k^3\sigma)}\,dk = \Delta^{n-\frac{1}{2}}\int_0^\infty A_\pm(k_1\Delta^{n-\frac{1}{2}})\,e^{i\Delta^{-\frac{1}{2}}(k_1\xi+\frac{1}{6}k_1^3T)}\,dk_1,$$

and we can note that $\frac{1}{3} < n < \frac{1}{2}$. Now as $\Delta \to 0$, for the case $\xi < 0$, we have a point of stationary phase where

$$k_1 = \left(\frac{-2\xi}{T}\right)^{\frac{1}{2}},$$

and it is clear that the dominant behaviour for $I$ cannot allow (5.23) to match with the term $(-\xi\tau)^{-\frac{1}{2}} = (-\Delta\xi T)^{-\frac{1}{2}}$ in (5.6) unless $A_- \sim \hat{A}_-k^{-\frac{1}{2}}$ as $k \to \infty$, where $\hat{A}_-$ is a constant. Further, the matching is not complete if the stationary phase contribution alone is used. A contribution (of the same order) is required from the singular point at $k = 0$ if $A_-$ behaves like $k^{-\frac{1}{2}}$ as $k \to 0$; in other words matching is possible if

$$A_-(k) \equiv \hat{A}_-k^{-\frac{1}{2}}.$$

The constant $\hat{A}_-$ can now be found by matching in detail, using the contributions both from the point of stationary phase (which is trigonometric), and from the singular point at $k = 0$ (which gives an algebraic contribution). Using both the real and imaginary parts of $I$, and ensuring that the purely algebraic terms are eliminated, we find that (5.23) matches precisely if written as

$$\hat{H} = \frac{a}{2\pi^{\frac{1}{2}}}T^{-\frac{1}{2}}\int_0^\infty k^{-\frac{1}{2}}\cos\left(k\chi + \frac{k^3}{6}\sigma + \frac{\pi}{4}\right)dk, \quad \chi < 0. \tag{5.24}$$

(Thus both $\hat{A}_-$ and the arbitrary constant $\theta_0$ are chosen.)

The technique for $\xi > 0$ is rather similar, although we now have a saddle in the range of integration together with the singularity at $k = 0$. The choice of $A_+(k)$ is just $\hat{A}_+k^{-\frac{1}{2}}$ and then the form of (5.23) for matching becomes

$$\hat{H} = \frac{b}{2\pi^{\frac{1}{2}}}\frac{1}{\sqrt{T}}\int_0^\infty k^{-\frac{1}{2}}\cos\left(k\chi + \frac{k^3}{6}\sigma + \frac{\pi}{4}\right)dk, \quad \chi < 0, \tag{5.25}$$

where $b$ occurs in (5.7). Clearly (5.24) and (5.25) are identical if $a = b$, but this can arise only for vanishingly small amplitudes and then the problem is rather trivial in that a linear far-field wave is matched to a linear near-field wave. Here, solutions (5.24) and (5.25) are still separated by the fully non linear similarity solution which is at $T = O(1)$ in a region $\xi = O(\Delta^{\frac{1}{3}})$. A diagrammatic representation of this situation will be given later (in figure 3) when the final stage of the matching to a near-field region has been completed.

The solutions given in (5.24) and (5.25) are multiple-scale representations valid where $T = O(1)$, and therefore these cannot be uniformly valid when the distance ($\xi$) and time ($T$) scales are of the same size. In terms of $n$, this occurs when both $\xi$ and $T$ are $O(\Delta^n)$ or, equivalently, when $R$ and $T$ are $O(\Delta^n)$. This region is still outside the similarity solution (which is where $\xi = O(\Delta^{\frac{1}{3}(n+1)})$ if $T = O(\Delta^n)$) provided $n < \frac{1}{2}$. Thus we introduce

$$R = \Delta^n R', \quad T = \Delta^n T' \quad (\tfrac{1}{3} < n < \tfrac{1}{2}),$$

whence, upon scaling $\eta$ and $\phi$, it easily follows that to leading order

$$H'_{T'T'} - \left(H'_{R'R'} + \frac{1}{R'} H_{R'}\right) = 0, \quad \hat{H} = \Delta^{-n} H', \tag{5.26}$$

the linear concentric wave equation, so if we consider the solution which is initially zero everywhere and which also is singularity free as $R' \to 0$ (Miles 1978*a*), then

$$H' = \int_0^\infty A'(k) \sin(kT') J_0(kR')\, dk. \tag{5.27}$$

This solution is to match with (5.24) or (5.25), depending on whether $R' - T' < 0$, $R' - T' > 0$, respectively. Writing (5.27) in the original variables $R$, $T$ and letting $\Delta \to 0$ we see that the contribution to the integral from the region where $kR/\Delta^n = O(1)$ is $O(\Delta^n)$ but from the rest of the range it is $O(\Delta^{\frac{1}{2}n})$. Thus the dominant contribution can be obtained by expanding $J_0$ for large argument, always provided that $A'(k)$ is nowhere singular. Thus from (5.27) we find that

$$H' \sim \left(\frac{2\Delta^n}{\pi R}\right)^{\frac{1}{2}} \int_0^\infty \frac{A'(k)}{\sqrt{k}} \sin\left(\frac{kT}{\Delta^n}\right) \cos\left(\frac{kR}{\Delta^n} - \frac{\pi}{4}\right) dk,$$

whence matching to the region where $R - T = \xi = O(\Delta^n)$, $T$ and $R+T$ large, requires only the term in $\cos(k\xi/\Delta^n + \frac{1}{4}\pi)$ and then $A'(k)$ is just a constant,

$$A' = \frac{a}{\pi\sqrt{2}} \quad (\xi < 0); \quad A' = \frac{b}{\pi\sqrt{2}} \quad (\xi > 0). \tag{5.28}$$

(As Miles points out, the solution (5.27) with $A'$ given by (5.28) and $a \sim b$ corresponds to the problem of linear concentric waves emanating from a point source at $R = 0$; the volumetric flow rate of the source is proportional to $a$.) In our presentation of the problem, the matching is to the fully nonlinear solution and also the solution (5.27) is not valid at the origin. This is because when $n = \frac{1}{2}$, the scalings used to define $R'$ and $T'$ recover the full equations, (2.1)–(2.3), with all terms of comparable size. Thus solution (5.27) is to be matched to the full problem in a neighbourhood $O(\Delta^{\frac{1}{2}})$ of the origin; this we cannot do analytically.

A diagrammatic representation of the various regions of validity is given in figure 3,

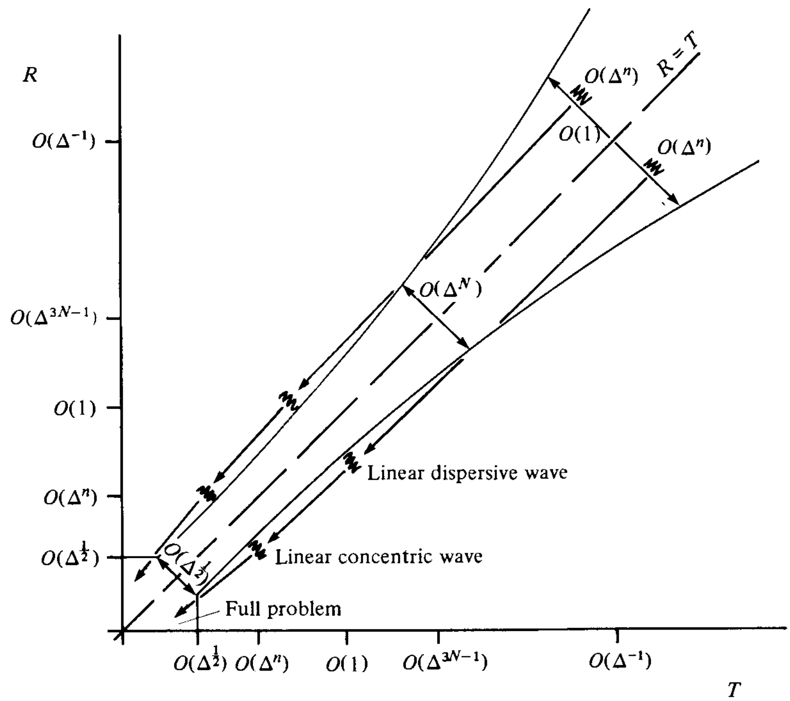


FIGURE 3. Regions of validity for the matching of the similarity solutions of the concentric KdV equation: the matching is indicated by the arrows and the similarity solution exists in the region $O(\Delta^N)$ about $R = T$.

where we have used the interpretation developed here (although others are possible). The matching is indicated as being from a region of the non-linear similarity solution into two linear regimes and thence into the full problem near the origin. Two such paths are shown, one from ahead and one from behind – but both always separated by – the similarity solution: the two matched solutions come together in the same region at the origin. The complexity of figure 3 should be compared with the simplicity of figure 1, where $\xi = O(1)$ in all regions.

We can now consider the matching of our second similarity solution, (5.10), with asymptotic behaviours (5.12) and (5.13). The procedure is exactly as given above, leading in the first instance to solution (5.23) which must be matched to the new solution. It is clear, however, that there is an important difference here, particularly when $\xi > 0$; the singularity at $k = 0$ produces exactly the required algebraic behaviour and so it is the trigonometric contribution which must vanish in this case. The solution which corresponds to (5.24) and (5.25) is then given by

$$\hat{H} = \pm\frac{2}{3}\left(\frac{2}{\pi T}\right)^{\frac{1}{2}}\int_0^\infty k^{-\frac{1}{2}}\cos\left(k\chi+\frac{k^3}{6}\sigma-\frac{\pi}{4}\right)dk, \tag{5.29}$$

where the ordering of the signs is as $\chi < 0$, $\chi > 0$. Two important aspects of this solution are immediately evident. First, there is no amplitude parameter which allows the solution for both $\chi < 0$ and $\chi > 0$ to correspond: this is due to the essential nonlinearity of the similarity solution [cf. $a \sim b$ in (5.24) and (5.25)]. Also, the different combination of dominant terms which gives rise to the correct matching requires the arbitrary constant, $\theta_0$, to be $-\frac{1}{4}\pi$ [again cf. (5.24) and (5.25)]. In fact this latter point shows that the matching to the concentric wave problem (5.26) now requires the $Y_0$ Bessel function. The matching does follow as before, even though $Y_0$ diverges as $R' \to 0$, since the dominant contribution still occurs for $R' \to \infty$; thus corresponding to (5.27) we find that

$$H' = \pm\frac{4}{3}\int_0^\infty \sin(kT')\,Y_0(kR')\,dk. \tag{5.30}$$

Finally, in a region $O(\Delta^{\frac{1}{2}})$ about the origin we obtain the full problem and then the various regions are exactly as depicted in figure 3.

To conclude this discussion, we turn our attention to the similarity solution of the plane (classical) KdV equation given by (5.1), with (5.14), (5.15) and (5.16), and which has the asymptotic behaviours (5.17), (5.18). We can use the same prescription as for the concentric equation to examine the solution in the linear régimes, that is, we introduce $\chi$ and $\sigma$ as defined in (5.20) but with $\alpha$ replacing $\Delta$. However, it is easy to see that if we seek a multiple-scale solution in terms of $\chi$, $\sigma$ and $T$ [where $T$ is now given by (4.1)], dependence on $T$ is necessarily absent and then we obtain the original KdV equation (if $n < \frac{1}{2}$). When $n = \frac{1}{2}$, the full equations defining the water-wave problem are once again recovered, with no simplification available. In other words, there is no asymptotic region in which a linear problem exists. That this should be the case for the plane KdV equation is really not so surprising. The reason can be traced to the difference in amplitude variation for the concentric KdV equation. In this latter equation, the amplitude varies according as $\chi$ and $T$ vary, and the $T$ dependence arises solely from the geometric effect. Of course, this is absent in the plane KdV equation and so the amplitude is governed by the scaling associated with $\chi$ (and $\sigma$) only, which constitutes an invariant transformation of the plane KdV equation. The sketch of the various regions is therefore similar to that given in figure 3 provided that the matching 'arrows' are deleted; we have just the region of validity of the similarity solution emanating from an $O(\alpha^{\frac{1}{2}})$ neighbourhood of the origin.

Although there is no *formal* asymptotic region where a linear problem can be written down, it would be most instructive if we could – in some sense – compare the plane and concentric KdV equations. The one avenue open to us is to employ the device adopted by Miles (1978*a*): we match in the linear limit of the plane KdV similarity solution, i.e. $A \sim B$. This will enable us to compare the two similarity solutions on an equal footing. (Naturally, we must use only (5.26) or (5.27), with $a \sim b$, for comparison; solution (5.29) is inadmissable.) The argument follows precisely that given above; if

$$\chi = \xi\epsilon^{-n}, \quad \sigma = T\epsilon^{1-3n}, \quad n < \tfrac{1}{2},$$

then

$$\hat{\eta} = \frac{2^{\frac{1}{3}}A}{\pi}\int_0^\infty k\sin(k\chi + \tfrac{1}{6}k^3\sigma)\,dk, \quad \hat{\eta} = \epsilon^{2n}\eta, \tag{5.31}$$

matches precisely with (5.17) and (5.18) if $A \sim B$. If $X$ and $T$ are the same size, i.e. $O(\alpha^n)$, then the solution is an arbitrary function of $(X-T)\alpha^{-n} = \chi$ which clearly matches with (5.31) as $\sigma \to 0$; in fact, this means that (5.31) is uniformly valid as $\sigma \to 0$. As we have already found, such a simple conclusion does not apply to (5.26) or (5.27) even when $a \sim b$. Now setting $\sigma = 0$ in (5.31), we see that the solution in an $O(\alpha^n)$ neighbourhood of the origin is proportional to the generalized function $\delta'(.)$

propagating to the right: this is the analogue of the corresponding problem for the concentric equation.

## 7. Discussion

The classical problem in water-wave theory has been examined from a point of view which takes as its central theme the existence of various KdV-like equations. Four such equations have been obtained: two expressed in Cartesian co-ordinates and two in plane polar co-ordinates. The main thrust of the work has then been to examine these equations (and their solutions) in terms of the underlying water-wave problem. This enables rather elementary transformations to be written down which show, for example, that for a certain class of solution only the plane and concentric equations are relevant.

The essential idea associated with the relation between the various KdV equations and the water-wave problem resolves itself into a discussion of the initial value problem. That is, given the classical equations for surface waves written using an appropriate non-dimensional scheme, what type of initial profile is required to generate a certain solution of a KdV equation? The answer to this question must be obtained in two stages: first, the relevant scaling of the near-field region is to be determined in terms of the scalings which define the KdV equation. Actually, the form of solution itself, for example similarity, may suggest the scalings, and then the second stage is quite evidently to choose the correct functional form of the initial disturbance. In the case of the non-similarity solutions this is quite straightforward for, as $T \to 0$, the near-field problem is substantially independent of $T$. This is precisely true for both the plane and nearly-plane equations, and if the geometric term $T^{-\frac{1}{2}}$ is ignored in the solution of the concentric equation, it is true also for this problem. However, the equation which we have dubbed the nearly-concentric KdV equation is not so easily managed. It was mentioned that for this equation the near-field behaviour had to be represented in terms of a rather special exponential decay as $T \to 0$: this particular avenue was not pursued in detail for our main interest was in the similarity solutions.

If similarity solutions only are considered then we can limit our discussion to the plane (classical) KdV equation, and the concentric KdV equation. In fact, it turned-out that the latter equation was by far the most rewarding. Extending the work of Miles (1978*a*), we have seen how the matching problem can be described in various regions of the physical plane. Also, the similarity solution itself was defined in a region which emanated from a neighbourhood of the origin where the full problem arose. In view of this the matching problem could not be extended back to the origin, and also the matching conditions had to be applied on one side or the other of the similarity solution. Nevertheless, our results agree in their essentials with the work of Miles if we consider that similarity solution which has a free constant (an amplitude) and let the amplitude tend to zero. A typical solution of this type was described, as was the new solution which decayed algebraically ahead of the wave front (see figure 2). This second solution was completely determined, but in other respects the matching problem was very similar.

If we compare the two similarity solutions in the linear régimes then we find that they each match onto one of the two available solutions of the linear concentric wave equation. The solution which decays exponentially ahead of the wave matches to the

$J_0$ Bessel function, whereas the solution with algebraic decay requires the $Y_0$ Bessel function. Both these solutions are then to match onto the appropriate solutions of the full equations sufficiently close to the origin. Of course, it had been our intention to characterize the various problems in terms of the precise initial data required for each one. This we cannot satisfactorily complete unless we allow the amplitude – where possible – to tend to zero. Such a procedure destroys the essential nonlinearity of the solution, but for one similarity solution of the concentric KdV equation it does allow a comparison with the corresponding solution of the plane KdV equation.

The classical KdV equation has a well documented similarity solution going back at least as far as 1964 (Berezin & Karpman). We have shown that for the water-wave equations this solution cannot be (formally) matched to a suitable linear problem unless the amplitude is small. Also, the same difficulty over the full equations being valid in a neighbourhood of the origin further complicates the issue. Certainly, for small amplitudes, both solutions are generated by suitable source-like behaviour at the origin; this is the usual way in which similarity solutions are produced in a whole range of mathematical problems.

In conclusion, we can expect that other physical situations which give rise to KdV equations – or other evolution equations from inverse scattering theory – can be described in the manner outlined here. In fact there may be other similarity solutions available for the water-wave problem, although none are known to date. [The inverse scattering theory for the plane KdV equation does not seem to allow a similarity solution in closed form (see Ablowitz & Segur 1977*b*).] Similarity solutions and the corresponding matching problems would appear to be a field worthy of further study.

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